Responding to Children's Mathematical Thinking in the Moment: An Emerging Framework of Teaching Moves

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Abstract: We build on earlier work to present an emerging framework of teaching moves used in one-on-one interactions to support and extend children's mathematical thinking. The framework identifies categories of teaching moves, rather than specific comments or questions, because how teachers enact a category depends on the situation. We illustrate the framework using examples from a case study of a highly skilled teacher's interactions with children engaging with fraction story problems in one-on-one interviews and class lessons. The teacher was selected because of her expertise in responsive teaching—teaching in which the instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking instead of being determined in advance. We discuss four major categories of teaching moves: (a) ensuring that the child is making sense of the story problem, (b) exploring details of the child's existing strategy, (c) encouraging the child to consider other strategies, and (d) inviting the child to generate symbolic notation. Our findings also highlight the potential usefulness of one-on-one interviews for both professional developers and researchers as well as the need for increased attention to the part of class lessons in which teachers circulate and engage in one-on-one conversations with children.

Keywords: responsive teaching, teaching practices, children's thinking, elementary school, fractions, teacher learning

A growing trend in research on teaching is to decompose what teachers do while they interact with students and subject matter (Ball and Cohen 1999; Grossman et al. 2009). One main purpose of this decomposition is to identify components of practice that can be discussed and rehearsed by teachers who want to improve their practice. In this article, we focus on identifying components of practice that support a particular vision of teaching, one that is centered on and responsive to children's mathematical thinking (Ball et al. 2001; Bobis et al. 2005; Kazemi et al. 2009).

We conceptualize *responsive teaching*[1] as a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking instead of being determined in advance. Thus, teachers must, in the moment, actively engage with children's thinking without imposing their own thinking and while also respecting the discipline of mathematics (Ball 1993; Lampert 2001). The interactive and spontaneous nature of this type of teaching makes it particularly challenging to enact.

We highlight what we consider to be a parsimonious set of elements that are necessary but not sufficient for teachers to provide opportunities for children to advance their mathematical thinking. Specifically, our model of responsive teaching consists of knowledge of children's mathematical thinking and two generative instructional practices that happen in the moment (Figure 1).

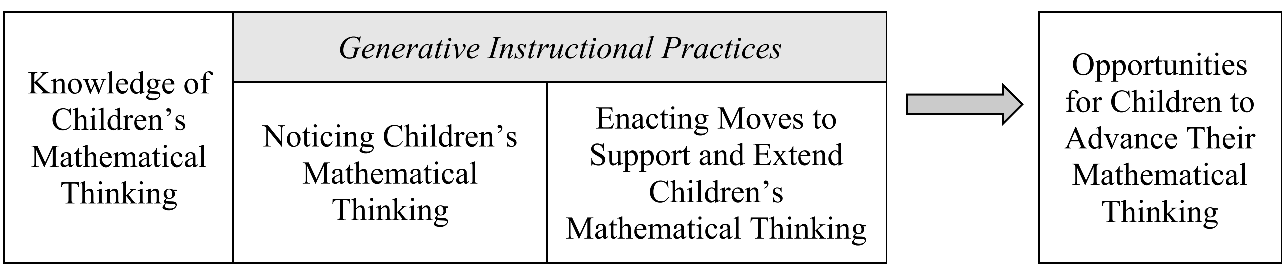

Figure 1. Model of teaching that is responsive to children's mathematical thinking.

We conceptualize knowledge of children's mathematical thinking as usable by teachers while they engage in instruction. Linked to a growing research base on children's ways of reasoning (see Lester 2007 for an overview), this knowledge includes organized frameworks for making sense of children's thinking and benchmarks for recognizing significant advancement of this thinking (see Empson and Levi 2011 for an example with children's fractional thinking).

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[1] We recognize that teachers can be responsive to students in a class setting in many productive ways. In this article, we use *responsive teaching* to refer to the part of teaching that is responsive to children's mathematical thinking.

We conceptualize the two generative instructional practices—noticing children's mathematical thinking and enacting moves to support and extend children's mathematical thinking—as domain specific in that they rely on specific knowledge of a target mathematical domain and learning in that domain. Noticing children's mathematical thinking involves attending to the details in children's strategies, interpreting what understandings are reflected in those details, and then deciding what to do next on the basis of those understandings (Jacobs et al. 2010). The practice of noticing occurs prior to the teachers' actual responses and as such is not observable. However, teachers' noticing is foundational to the observable moves that teachers enact to support and extend children's mathematical thinking.

We conceptualize a *teaching move* as a unit of teaching activity that has coherence with respect to a purpose. In prior work, we identified a set of eight categories of responsive-teaching moves based on an analysis of 65 teachers' one-on-one interviews with 231 children (Jacobs and Ambrose 2008). We found that identifying categories of teaching moves that enabled children to work from what they understood to make sense of the mathematics—rather than identifying specific comments or questions—better reflected the situational nature of these interactions (Jacobs and Ambrose 2008; Jacobs et al. 2011). In this article, through an analysis of the teaching moves of a highly-skilled teacher, we share an emerging framework that builds on this work and begins to extend it in two ways.

First, this study was focused on a different mathematical domain. Initial work focused on children in grades K–3 (approximately 5–8 years of age) engaged with whole-number story problems, and the current study was focused on children in grades 4–5 (approximately 9–10 years of age) engaged with fraction story problems. Different content involves different mathematical opportunities and challenges in terms of the mathematics itself, the cognitive and social resources children bring to instruction, children's engagement with the mathematics, and teachers' comfort levels with the mathematics. Compared to whole numbers, fractions have traditionally been challenging for many students and teachers, both in terms of developing conceptual understanding and facility with operations (Stafylidou and Vosniadou 2004). In this case study, we began to explore consistency of categories of teaching moves across mathematical domains.

Second, this study included an additional instructional setting. Prior work was focused on interviews, and the current study included both interviews and class lessons. Interviews are one-on-one interactions—outside the class setting—in which a teacher has focused opportunities to learn about and advance a child's mathematical thinking during problem solving. We recognize that interview settings are less complex than class settings, but in both cases, teachers must observe, listen, and then, in the moment, determine and offer comments and questions. Thus we argue that the same expertise—knowledge of children's mathematical thinking, noticing of children's mathematical thinking, and enacting moves that support and extend children's mathematical thinking—is needed in both settings. In this case study, we began to explore the applicability of the framework in class settings and to test our assumption of the overlap of teaching moves in interview and class settings.

1 Methods

Our main goal is to present an emerging framework that characterizes teaching moves that are responsive to children's mathematical thinking. The construction of the framework was informed by prior work (Jacobs and Ambrose 2008) and extended by the current study focused on teaching moves enacted by a highly-skilled teacher in both interviews and class lessons involving fractions. The teacher was one of 7 case-study participants in a small expert-teaching study in which we analyzed a snapshot of the practices of teachers who were already skilled at responding to children's mathematical thinking. These teachers had previously engaged in professional development on children's thinking, but no professional development was involved in this study. Instead, our goal was to learn from teachers who already possessed substantial expertise.

1.1 Participants

In this case study, we focused on the teaching of Annie Keith in a multiage class of fourth and fifth graders with 15 boys and 10 girls. She worked in an inclusion class in which she cotaught with a teacher who had special expertise in working with children with learning, emotional, or behavioral disabilities. Ms. Keith was purposefully selected because of her expertise in responsive teaching; we chose to begin our exploration at the highly skilled end of the responsive-teaching continuum to better understand how this type of teaching could be enacted and to collect clear illustrations of responsive-teaching moves.

At the time of the study, Ms. Keith had been an elementary school teacher for 29 years, teaching multiage classes of fourth and fifth graders the last 8 years. For 28 of the 29 years, she has been connected with the Cognitively Guided Instruction (CGI) project and the exploration of children's mathematical thinking (Carpenter et al. 2015; Carpenter et al. 2003; Empson and Levi 2011). CGI is a research and professional development project in which teachers gain access to research-based knowledge about children's mathematical thinking and design instruction to build on children's thinking. Ms. Keith had participated in multiple years of CGI professional development, possessed extensive knowledge of children's mathematical thinking, focused on children's thinking in her teaching, and helped other teachers learn about children's thinking and its role in instruction.

Ms. Keith's instructional interactions are conducive to studying teaching moves that are responsive to children's mathematical thinking because, in her class, children's thinking is valued and visible during problem solving. Rather than demonstrating strategies, Ms. Keith encourages children to generate and use strategies that make sense to them, and she routinely elicits and builds on their ideas. Children also regularly learn from one another because they are expected to explain and justify their thinking.

Ms. Keith teaches in a small city in the midwestern region of the United States and, at the time of our study, her school served a diverse group of 471 students (39% White, 21% African American, 17% Asian, 14% Hispanic, and 9% Other). Twenty percent of the students were classified as English Language Learners and 43% as economically disadvantaged.

1.2 Data Sources

Data for this article, collected during two consecutive days near the end of the school year, were drawn from two class lessons and one-on-one interviews with 5 children. All class lessons and interviews were video recorded and field notes were taken during the events. Although both story problems and equations were posed in the class lessons and interviews, we chose to focus only on story problems because they were the focus of the original work (Jacobs and Ambrose 2008) and are particularly facilitative for children to work from what they know about partitioning and combining quantities. In future work, we will explore teaching moves with equations. Table 1 provides a summary of the story problems discussed in this article.

Table 1

Story Problems Discussed During the Class Lessons and Interviews

Event	Problems
Lesson 1	Kenzie loves to go on long hikes. She often hikes with her college friends. She knows it is important to drink water when she hikes. She drinks ___ cup of water for every mile she hikes. Her water bottle holds 4 cups of water. How many miles can she hike before her water runs out? (Children chose 1/2, 3/4, or 2/3 to replace the blank.)
Lesson 2	8 people want to share 13 sandwiches so that they all have the same amount of sandwich. How much sandwich should each person get? 8 children are sharing 5 hamburgers equally. How much hamburger does one child get?
Interviews with individual children	8 children want to share 14 cookies so that each child gets the same amount. How much can each child get? Meleri is making some cookies. It takes 1 1/2 cups of brown sugar to make each batch. If Meleri has 10 cups of brown sugar, how many batches of cookies can she make? Ms. Hughes is making lemonade for the class party. She has 8 cups of sugar. She needs 2/3 cup of sugar to make a pitcher of lemonade. How many pitchers of lemonade can she make? (alternative number: 3/4 cup of sugar) During art class, the teacher gave a table of 8 students 6 bars of clay to share equally. How many bars of clay does she need for the remaining 12 students if she wants to be sure that each person gets exactly the same amount?

1.2.1 Class lessons. For the two class lessons, Ms. Keith was asked to focus on fractions but was otherwise free to choose the most appropriate problems or activities for her students. Her first class lesson lasted 72 minutes and had four main parts: (a) whole-group warm-up in which the children engaged with a fraction-computation puzzle (18 minutes); (b) whole-group discussion in which a child shared what she had realized about fractions and divisions on a ruler (12 minutes); (c) individual problem solving, with Ms. Keith circulating to converse with children about the hiking problem involving division of a whole number by a fraction, after which children could choose to solve another story problem or complete some fraction equations (24 minutes); and (d) whole-class discussion about the hiking problem from the problem-solving segment (18 minutes).

The second class lesson lasted 80 minutes and had three main parts: (a) whole-group discussion in which a child shared what she had realized about equivalent fractions (30 minutes); (b) individual problem solving, with Ms. Keith circulating to converse with children about two equal-sharing problems with fractional answers, after which children were asked to write their own story problem with at least one fraction or mixed number (35 minutes); and (c) pairs of children sharing strategies and problems they had created during the problem-solving segment (15 minutes).

In this article, we focus on the problem-solving segments in which Ms. Keith circulated to converse with individual children about story problems. Across both lessons, this segment was between one third and one half of the total class time. During the circulation time in the first lesson, Ms. Keith worked with 11 children—7 children for less than 1 minute each, 2 children for 2–4 minutes each, and 2 children for 4–6 minutes each. During the circulation time in the second lesson, she worked with 7 children—5 children for about 2 minutes each and 2 children for 9–11 minutes each. Some of the longer conversations occurred in two parts, with Ms. Keith giving the child an opportunity to work independently before continuing the conversation.

1.2.2 Interviews. For the interviews, Ms. Keith was asked to select students with a range of achievement levels and to work with them individually to learn about their thinking. She was provided a list of potential problems focused on fractions but was free to add, delete, and adapt problems so that the interview was appropriate for each child. Ms. Keith individually interviewed 5 children, with three interviews lasting 45 minutes, one lasting 30 minutes, and the final one lasting 15 minutes. Each child solved at least one story problem, and 7 story problems were posed across the 5 children. The conversations about each story problem ranged from 5 to 25 minutes with an average of 13 minutes.

1.3 Analyses

We analyzed the class and interview video data to characterize the teaching moves that Ms. Keith used to be responsive to her students' mathematical thinking. Our emerging framework is designed to be at a grain size that is useful to researchers who want to understand and capture responsive-teaching expertise and to teachers who want to draw on the categories of moves as possible options (not a prescriptive checklist) during real-time interactions. To ensure that teachers could access this framework in the moment, we needed to constrain the total number of categories.

Identifying categories of teaching moves required consideration of each teaching move *in relation to* children's thinking in a particular situation because the same question may be responsive in one situation but not in another. Thus, analysis involved not only consideration of what the teacher said and did but also of the situation, including what the child said and did before and after the teaching move. Further, because we conceptualized a teaching move as a unit of teaching activity that has coherence with respect to a purpose, we were not coding individual talk turns. Instead, we looked for conceptual breaks in the conversation to determine our unit of analysis (Jacobs and Morita 2002). Sometimes the unit consisted of a teacher's single comment or question, and other times, it included a linked sequence of comments and questions.

To connect with our prior work (Jacobs and Ambrose 2008), we began our analysis of Ms. Keith's data by focusing on the interviews and using the eight categories of teaching moves previously identified—four that occurred before the correct answer was given and four that occurred after (see the left-hand side of Figure 2). Distinguishing moves before and after the correct answer was given was intended to highlight the underutilized potential of mathematical conversations following a correct answer (see also, Fraivillig et al. 1999). In Ms. Keith's interviews, we tracked the eight categories of teaching moves with links to specific examples so that the research team could discuss the appropriateness of these categorizations. We also looked for other teaching moves—not reflected in the framework—that we felt contributed to the responsive nature of these interactions. We identified no additional major categories. However, for the two categories specifically focused on the child's existing work (*exploring what the child has already done* and *promoting reflection on the strategy the child has just completed*), we found five subcategories that we felt merited attention because of the potential usefulness of additional specificity for both researchers and teachers.[2] Rather than substantially expand the number of categories by adding five subcategories to each of the two major categories, we chose to collapse the categories of moves before and after the correct answer into a single framework by capitalizing on the many parallels between moves that occur before and after the correct answer is given (see the right-hand side of Figure 2). This collapsing enabled the inclusion of a single set of five subcategories (to be described in the findings section) while maintaining a reasonable overall number of categories so that teachers could access the framework, in the moment, when interacting with students.

[2] Our ultimate goal is to generate a framework that applies to both whole-number and fraction problem solving, and we believe that all the subcategories generated here would also apply in whole-number situations, but this assumption merits further study.

We then applied this collapsed framework to Ms. Keith's one-on-one interactions in class lessons when she circulated during the problem-solving segment, using the same process to look for any needed adjustments. Note that although we no longer formally distinguished teaching moves before and after the correct answer was given, we analyzed teaching moves at both time points. We found that adjustments were not needed, and the categories of teaching moves in the interviews adequately captured the categories of teaching moves in the class lessons.

Jacobs and Ambrose (2008) Analysis Original Categories of Responsive-Teaching Moves Used in One-on-One Interactions	
Before a correct answer is given	*After a correct answer is given*
Ensuring that the child understands the problem	
Exploring what the child has already done	Promoting reflection on the strategy the child just completed
Reminding the child to use other strategies	Encouraging the child to explore multiple strategies and their connections
	Connecting the child's thinking to symbolic notation
Changing the mathematics in the problem to match the child's level of understanding	Generating follow-up problems linked to the problem the child just completed

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Current Analysis Emerging Framework of Responsive-Teaching Moves Used in One-on-One Interactions
Ensuring that the child is making sense of the story problem
Exploring details of the child's existing strategies
Encouraging the child to consider other strategies
Inviting the child to generate symbolic notation
Adjusting the problem to match the child's understandings*

* This category was not evident in substantial ways in the data shared in this article, but we include it for completeness. We also know that this category is visible in a larger sample of Ms. Keith's interactions because we saw evidence of this move with equation work. However, equations are outside the scope of this article, and thus we will not discuss this framework category further.

Figure 2. Linking our emerging framework of responsive-teaching moves used in one-on-one interactions to the original categorization.

2 Findings

We found that Ms. Keith was responsive to children's thinking in multiple ways, and underlying most of these moves was an informed spontaneity; her moves could not have been scripted in advance because they involved adapting instruction on the basis of what she noticed a child say or do in a particular situation. Thus, our analysis of her interactions was an attempt to bring structure to a set of skilled moves that were situation based. In this section, we describe the framework of responsive teaching moves that emerged, and for each category, we share at least two examples—one from an interview and one from a class lesson—to illustrate how these categories of teaching moves were consistent across instructional settings. (See Figure 3 for an overview of the categories linked to examples discussed in the findings section.)

Categories of Responsive-Teaching Moves		Examples
Ensuring that the child is making sense of the story problem		• What are you thinking when you read this problem—what is the picture in your head? • She drinks 1/2 cup of water for every mile she hikes … does that change how you are thinking about things?
Exploring details of the child's existing strategy	*Posing starter questions to the child*	• Wow, you did a lot of nice thinking here. Can you tell me what you were doing? What were you thinking about first? • Walk me through what you thought about.
	Pressing the child for a detailed explanation of his or her problem-solving process	• So I noticed you cut this [cookie] into eighths, why did you choose to cut that one into eighths? • Here you made this model [points to the first partitioned cookie], how come you didn't go ahead and make the lines on there [points to the 5 cookies that were not partitioned]?
	Probing the child's representation to highlight connections to the problem context	• It says it takes 1 1/2 cups of brown sugar to make each batch. Where on your grid here does it show 1 1/2 cups of brown sugar? • [Points to the shaded 3/4] And how far has she gone when she drinks that much?
	Questioning the child about quantities and their relationships in the strategy	• [In comparison to the original problem involving 2/3 cup every mile] If she drank 1 cup of water for every mile, how many miles would she go? ... Is 2/3 more or less than 1 whole cup? ... So do you think she'll be able to go more than 4 miles if she's drinking less water?" • $2/4 \div 2 = 2/4 \div 1/2$ True or False?
	Inviting the child to think ahead before executing a problem-solving step	• [Posed prior to solving] Would the children all get one hamburger? • [Posed in the midst of problem solving before the child had completed his drawing] So you're going ahead and drawing it, but you already know the answer?
Encouraging the child to consider other strategies		• [After the child has given three incorrect answers before giving the correct answer] Do you think there is a second way you could check … so you're like, "Okay, this makes sense *to me* what I'm doing." • Do you see a relationship between these two [strategies]?
Inviting the child to generate symbolic notation		• Do you want to write that [relationship] down so you are recording it as you are working? • [After the child has given a complete explanation of her strategy] When you go through and you think about what those steps were—what you said when you were talking out loud to me—do you think you could write the equation for what you were just saying?

Figure 3. Framework of responsive-teaching moves with examples that Ms. Keith used in one-on-one interactions in interviews and class lessons.

2.1 Ensuring That the Child is Making Sense of the Story Problem

A big challenge for children when solving story problems is understanding what is happening in the story and what question is being posed. Thus, the first major category of teaching moves involves ensuring that the child understands the problem, and these moves often, but not always, occur early in the interaction.

Ms. Keith regularly assessed children's understanding of the problem, highlighting the specific question that was being posed. Sometimes their discussion was focused on the child's understanding of the entire problem, such as in the interview when Jane[3] was unsure how to begin solving the problem about 8 children sharing 14 cookies: "What are you thinking when you read this problem—what is the picture in your head? Could you retell me the problem, like in your own words? What is it saying to you?"

Other times, Ms. Keith focused on specific parts of the problem (including vocabulary) that were challenging for a particular child. In the first class lesson, Rex was trying to figure out how many miles Kenzie could hike with 4 cups of water if she drank 1/2 cup of water every mile. When Ms. Keith arrived at his seat, Rex had determined that Kenzie could hike 2 miles because he had misinterpreted the problem to mean that she drank one half of her total amount of water every mile rather than 1/2 cup of water every mile. Ms. Keith recognized Rex's misinterpretation, and reread a strategic part of the problem: "She drinks one-half cup of water for every mile she hikes;" and then asked, "Does that change how you are thinking about things?" When Rex answered affirmatively, she asked why, and when they continued discussing the problem context, Rex asked, "What are cups?" further revealing the source of his confusion.

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[3] All children's names are pseudonyms.

After additional discussion including clarification of the vocabulary, Rex better understood the story and was able to solve the problem.

2.2 Exploring Details of the Child's Existing Strategy

The second major category of teaching moves captures the essence of being responsive to children's mathematical thinking because teachers must understand children's existing thinking before they can hope to build on it. Thus, we privilege this category by also highlighting five subcategories that Ms. Keith regularly demonstrated: (a) posing starter questions, (b) pressing for detailed explanations of the problem-solving process, (c) probing representations, (d) questioning quantities and their relationships, and (e) inviting the child to think ahead.

2.2.1 Posing starter questions to the child. To initiate conversations about children's ideas, teachers need to elicit what children have already thought about and done, even if their work is incomplete, only partially correct, or seems obvious to the teacher. Sometimes Ms. Keith began her conversations with general questions, such as for Alice in her interview: "Wow, you did a lot of nice thinking here. Can you tell me what you were doing? What were you thinking about first?" Similarly, in the first class lesson, Ms. Keith said to Amir, "Tell me what you did," and during the second class lesson, she requested of Jarvis, "Walk me through what you thought about." These general prompts were sufficient to indicate to the child that it was time to share how he or she had solved (or started to solve) the problem. At times, even Ms. Keith's arrival at a child's seat functioned as an implicit request for a child to begin explaining his or her thinking because of the class climate already created—one in which children's explanations were expected and valued.

We most often saw these starter questions after children reached an answer or were at a stopping point in the interviews, or when Ms. Keith first arrived at children's seats during class lessons. Although these starter questions effectively initiated conversations, they were less situation-specific and thus generally insufficient by themselves for supporting and extending children's thinking.

2.2.2 Pressing the child for a detailed explanation of his or her problem-solving process. Children's explanations of their strategies and ideas are often incomplete or procedural in nature (i.e., focused on the steps they performed rather than the reasons they performed them), and teachers' questions can help children make these explanations richer and more complete—a feature of instruction that has been positively linked to student achievement (Webb et al. 2014). We noted that Ms. Keith's inquiries focused on mathematically important steps in the problem solution (e.g., how children partitioned or how they combined fractional parts to arrive at a final answer) as well as points at which children exhibited uncertainty.

In her interview, Alice solved the problem about 8 children sharing 14 cookies (see Figure 4) by drawing 8 rectangles (children) and giving each child 1 cookie (marking "1" for each cookie in the rectangles). She tried to give them each another cookie (again, marking "1" for each cookie in the rectangles), but at the sixth child, stopped and crossed out the 6 cookies she had tried to distribute in the second round (the second "1" in the first six rectangles). She then drew 6 new rectangles (cookies) and partitioned the first rectangle into 8 parts, distributed 1/8 to each child by writing "1/8" in each rectangle representing a child, looked at her remaining 5 cookies (but did not partition them), and answered that each child would receive 1 6/8 cookies.

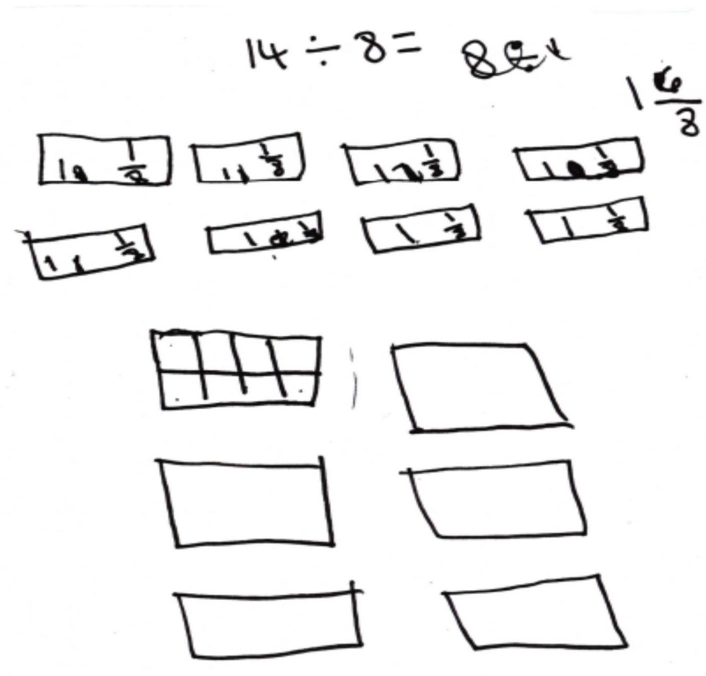

Figure 4. Alice's work for 8 children sharing 14 cookies.

Ms. Keith pressed Alice for three mathematically important details of her problem-solving process: (a) why she stopped passing out whole cookies at the sixth child ("I see you went on and then you stopped yourself here [points to the 6th child], why?"); (b) why she partitioned the first cookie into eighths ("So I noticed you cut this one into eighths [points to the partitioned cookie], why did you choose to cut that one into eighths?"); and (c) why she did not need to partition the final 5 cookies ("Here you made this model [points to the first partitioned cookie]; how come you didn't go ahead and make the lines on there?" [Points to the 5 cookies that were not partitioned]). Note that, unlike the starter questions in the previous category, these questions could not have been scripted in advance because they were focused on strategy details that Ms. Keith could have known only if she were paying close attention to Alice while she worked through the problem. In fact, each question Ms. Keith posed began with a mathematically important detail that she had noticed in Alice's strategy.

Conversations about the problem-solving process were sometimes less extensive in class interactions but were still focused on the mathematically important details. In the second class lesson, Hannah began explaining her strategy for solving the problem about 8 people sharing 13 sandwiches: she gave 1 sandwich to each person and then began distributing 1/2 of a sandwich at a time from the remaining 5 sandwiches. At this point, Ms. Keith interrupted her explanation to ask, “Why did you think a half?” Hannah responded, “Because one half of eight is four, and five is pretty close to it.” This clarification helped Ms. Keith learn that Hannah had made a reasoned choice for this step, in contrast to children who automatically move to halves after distributing wholes, without considering why (Empson and Levi 2011).

2.2.3 Probing the child’s representation to highlight connections to the problem context. A child’s representation of the problem situation—whether it involves pictures, tables, manipulatives, or numbers—is an important component of problem solving. Ms. Keith’s questions in this category focused on clarifying what problem quantities children were representing so that the strategy remained meaningful. At times, Ms. Keith highlighted a part of the problem context and asked where it could be found in a child’s representation. In her interview, Alice used a table to figure out how many batches of cookies Meleri could make with 10 cups of brown sugar if it took 1 1/2 cups of brown sugar to make each batch (see Figure 5). Ms. Keith asked her to locate parts of the problem context in her table representation [“grid”] with questions such as “Where are your batches?”; “Where’s your brown sugar?”; “It says it takes 1 1/2 cups of brown sugar to make each batch. Where on your grid here does it show 1 1/2 cups of brown sugar?”; and “Where on your grid here does it show 10 cups of brown sugar?”

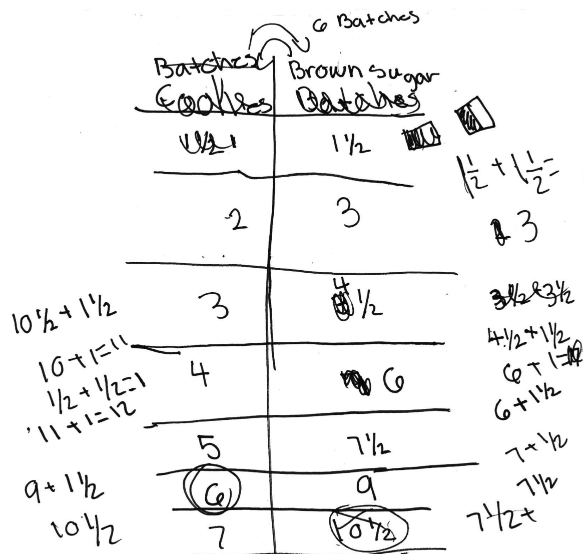

Figure 5. Alice’s table for determining the number of batches she could make from 10 cups of brown sugar if she uses 1 1/2 cups for each batch.

Other times, Ms. Keith started with the child’s representation and inquired about what a particular part of the representation meant in the problem context. In the first class lesson, Mae was trying to find the number of miles Kenzie could hike with 4 cups of water if she drank 3/4 cup of water every mile. When Ms. Keith arrived at her seat, Mae shared that she had done something wrong and that her picture (shown in Figure 6) was “not really working.” After rereading the problem with Mae, Ms. Keith focused on Mae’s existing representation and its connection to the problem context.

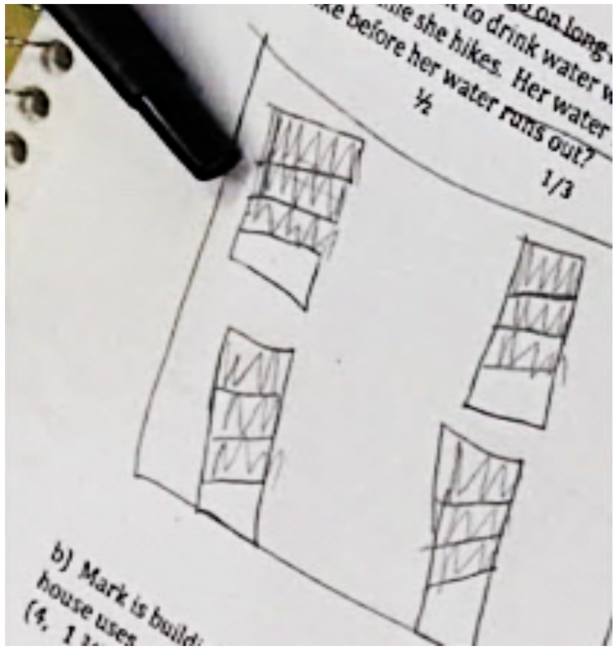

Figure 6. Mae's initial work for determining the number of miles one could hike with 4 cups of water if she uses 3/4 cup of water every mile.

[T]eacher: So tell me about your picture, what did you start off by thinking about?
[S]tudent: I drew four cups because it said, right here [points to the written problem] it says her water bottle holds four cups of water.
T: Okay
S: So I did the bottle—I did the four cups and then I put three fourths. So then I'm stuck because I don't get it.
T: So she drank this [points to the shaded 3/4 in the first cup]? You're saying that's how much of that cup she drank? [Mae agrees.] Would she have any left?
S: She'd have a fourth left.
T: She'd have a fourth left of that one cup, right? [Mae agrees.] And how far has she gone when she drinks that much [again points to the shaded 3/4 in the first cup]?
S: A mile
T: She's gone one mile.

After this clarification of how Mae's picture linked to quantities in the problem context, Ms. Keith encouraged Mae to augment her picture so that she could keep track of all the problem quantities. While Mae continued working, Ms. Keith was pulled away by some other children, but Mae was able to finish the strategy on her own, this time successfully keeping track of both the water and mile quantities (with a final answer of 5 miles and 1/4 cup of water left over).

2.2.4 Questioning the child about quantities and their relationships in the strategy. The ability to reason about relationships between quantities is a hallmark of understanding mathematics (Carpenter et al. 2003), and because children's problem-solving strategies involve operating on quantities, strategy discussions provide fertile ground for exploring two or more quantities and how they are related. At times, Ms. Keith highlighted relationships that were tightly linked to the problem context, such as in the first class lesson when Ethan was solving the problem about how many miles Kenzie could hike with 4 cups of water if she drank 2/3 cup every mile. When he gave his initial answer as 4 miles, Ms. Keith engaged him in an exploration of the relationship between the amount of water per mile and the number of miles hiked by asking, "If she drank *one* cup of water for every mile, how many miles would she go?" By highlighting a situation in which 4 miles would be the correct number of miles hiked, this question built on Ethan's answer of 4 miles and his likely (incorrect) strategy of equating a cup of water with a mile hiked. After determining that Kenzie could go 4 miles if she drank 1 cup of water for every mile, Ms. Keith asked, "Is two thirds more or less than one whole cup?" When Ethan indicated that he knew that 2/3 was less than 1, Ms. Keith followed up with, "So do you think she'll be able to go more than 4 miles then if she's drinking less water?" Throughout this conversation, Ms. Keith was highlighting the relationship between the quantities, building on Ethan's ideas, and enabling Ethan to work toward making sense of the mathematics.

Other times, Ms. Keith explored relationships among quantities more generally, using the child's strategy as a springboard to explore related ideas. She could not script these moments in advance but instead needed to recognize opportunities when they arose. In her interview, Gabby answered, "10 pitchers R 1/2" for the problem about how many pitchers of lemonade Ms. Hughes could make from 8 cups of sugar when it takes 3/4 cup of sugar for each pitcher. Ms.

Keith then engaged Gabby in an exploration of the 1/2 cup remainder and its relationship to 3/4 cup per pitcher. As part of the discussion, Gabby chose to explore a related idea: determining 1/2 of 2/4. When she wrote a number sentence to represent her thinking on this issue, she struggled with whether to write "2/4 ÷ 1/2" or "2/4 ÷ 2." Later in the conversation, Ms. Keith revisited Gabby's point of uncertainty by posing a true/false number sentence:

2/4 ÷ 2 = 2/4 ÷ 1/2   True or False?

This exploration of relationships among quantities went beyond the original problem but arose from Gabby's comments about her strategy. The conversation occurred only because Ms. Keith recognized and seized the opportunity to explore an important mathematical idea that children sometimes confuse (dividing in half vs. dividing by 1/2).

2.2.5 Inviting the child to think ahead before executing a problem-solving step. Children sometimes go through a problem-solving process in a rote or methodical manner, not taking advantage of what they already know. To counteract this tendency, Ms. Keith sometimes asked children at strategic points during their problem solving to anticipate the outcome of certain steps before executing them. In the second class lesson, Jarvis had successfully completed the first problem about 8 people sharing 13 sandwiches, and was ready to move on to the second problem about 8 children sharing 5 hamburgers. As Jarvis was beginning to solve the problem, Ms. Keith asked him to predict whether all children would get one hamburger (similar to how all people received one whole sandwich in the previous problem). Jarvis was able to think ahead and explained that they would not because with only 5 hamburgers and 8 children, "you've gotta split them up."

An invitation to think ahead was also issued to help children use what they knew to solve problems in more efficient ways. In his interview, Ethan was asked, but not required, to offer an answer before he completed his drawing of 8 students sharing 6 clay bars (see Figure 7). Ethan drew 8 circles for students and 6 squares for clay bars. He then drew 1/2 of a clay bar under each student and explained that he had used 4 clay bars and had 2 remaining, and that meant each student would get 1/4 more because "that would be one [bar] for 4 people and another one for 4 people." He began drawing 1/4 of a clay bar under each student, but Ms. Keith noted that he had essentially already given the answer and encouraged him to determine the answer without completing his drawing. During the following interaction, Ethan engaged with more advanced ideas while still continuing to draw (and eventually completing) 1/4 of a clay bar under each student.

T: So you're going ahead and drawing it, but you already know the answer?
S: I do?
T: Well, didn't you tell me how much this person would get (points to the first circle representing a student)?
S: A half and a fourth
T: Yeah, how much is the half and the fourth?
S: Three fourths. So that's the answer?
T: Well, is that this part that you are looking for (points to the part of the written problem where 8 students share 6 clay bars)?
S: Yeah

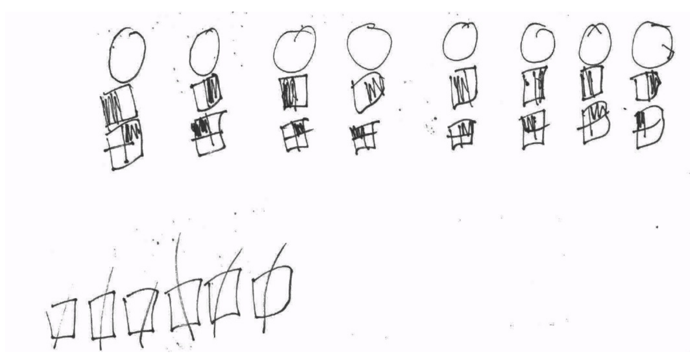

Figure 7. Ethan's work for 8 students sharing 6 clay bars.

Children may cling to inefficient strategies for a variety of reasons (e.g., incomplete conceptual understanding, assumption of teachers' expectations of elaborated written work, or lack of knowledge of how to record mental strategies), and asking them to think ahead can advance their thinking. However, it is critical that children—like Ethan in this example—have the option to reject the teachers' suggestions or engage in the more and less sophisticated thinking simultaneously. Otherwise, children lose control of the problem-solving process, and instead of their own thinking being supported and extended, the teachers' thinking becomes dominant.

2.3 Encouraging the Child to Consider Other Strategies

Flexibility in strategy use is an important characteristic of successful problem solving because not only do children learn alternative paths to use when they reach an impasse but also comparisons of strategies can be useful in highlighting important mathematical relationships (Heinze et al. 2009). However, children are often hesitant to consider alternative strategies, and thus the third major category of teaching moves involves encouraging children to consider other strategies.

Ms. Keith sometimes used the suggestion to consider other strategies to help children increase their confidence and ability to reach a successful solution in more than one way. In the first class lesson, Amir was trying to figure out how many miles Kenzie could hike with 4 cups of water if she drank 2/3 cup of water every mile. When Ms. Keith arrived at his seat, he had used several relationships that he knew ($3 \times 2/3 = 2$ and $2 \times 2 = 4$) to conclude that Kenzie could hike 12 miles because $3 + 3 + 3 + 3 = 12$. In sharing his strategy, he revised his answer three times, first to 18, then to 9, and finally to the correct answer of 6, but he was still not confident about how many threes he needed to add. Ms. Keith encouraged him to solve it another way so that it made sense *to him*:

> Do you think there is a second way you could check, because you've thought about 12, you've thought about 18, and then you thought 9 and now you're thinking, "Wait a minute"? Could you make a grid or some other way that you could think about solving it? I just want you to double check so you're like, "Okay this makes sense *to me* what I'm doing."

Ms. Keith also used the suggestion to consider other strategies to provide opportunities for comparing ways of reasoning, either within children's own problem solving or by comparing their strategies with those of other children. In the second class lesson, Alice and Sasha successfully solved the two related problems: 8 people sharing 13 sandwiches and 8 children sharing 5 hamburgers. Alice mentally determined that each person received one sandwich, then drew one of the five remaining sandwiches and partitioned it into eighths, and without drawing the rest of the sandwiches, determined that each person received 1 5/8 sandwiches. She solved the hamburger problem similarly, by drawing and partitioning only the first hamburger to determine that each child received 5/8 of a hamburger. Sasha described the same reasoning although he did not need to draw and partition the first item in each problem. Ms. Keith asked each child to reflect on how his or her own strategies on the two problems compared; Sasha was asked, "Do you see a relationship between these two [strategies]?" and Alice was asked, "Could you use part of what you were thinking up here [sandwich problem] to help you down here [hamburger problem]?" Sasha argued that the problems required essentially the same strategies, but Alice did not see the relationship and Ms. Keith did not force her to engage with mathematics that did not yet make sense to her. This contrast illustrates how the same move may unfold differently with different children, and it is not a correct answer or explanation that determines whether a move is responsive. On the contrary, we would consider both of these moves responsive because Ms. Keith prompted each child to build on existing work by exploring mathematical connections in ways that made sense to him or her.

2.4 Inviting the Child to Generate Symbolic Notation

The fourth major category of teaching moves highlights the important role of symbolic notation and building meaning for such notation in mathematical understanding. In contrast to previously discussed categories in which Ms. Keith questioned children about their existing representations, here she invited children to *generate* numerals, expressions, or equations that connected to other representations or ideas they had shared.

Ms. Keith sometimes made symbolic notation requests when the child was solving a problem mentally or with minimal recording. In Ethan's interview, he stated a relationship he knew ("two thirds times three is two") while determining how many pitchers of lemonade Ms. Hughes could make from 8 cups of sugar when it takes 2/3 cup of sugar for each pitcher. As a way to highlight that relationship, Ms. Keith suggested that he record it: "Do you want to write that [relationship] down so you are recording it as you are working?" Recording their ideas while working helps children view symbolic notation as a thinking tool (vs. only as a way of communicating a solution), and the resulting representation can also serve as a focal point for discussion of the details of their thinking.

Other times, Ms. Keith made symbolic notation requests to help children link their informal strategies to more formal notation to forge connections among multiple representations. In the second class lesson, Hannah explained how she had solved the problem about 8 people sharing 13 sandwiches by passing out 1 sandwich, then 1/2 of a sandwich, and finally 1/8 of a sandwich before combining the resulting quantities to answer 1 5/8 sandwiches per person (see Figure 8).

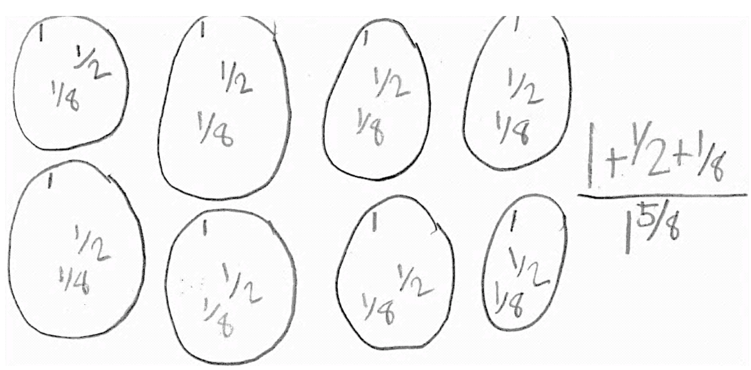

Figure 8. Hannah's work for 8 people sharing 13 sandwiches.

After Hannah gave a detailed explanation of her partitioning and combining the resulting quantities by recognizing that 1/2 was equivalent to 4/8, Ms. Keith built on her thinking by asking her to generate symbolic notation for her strategy:

> You had a great way of explaining it to me. And when you explained it to me, you also explained in mathematical talk when you were kind of like using equations. Can you write down those steps that you said with equations? ...When you go through and you think about what those steps were—what you said when you were talking out loud to me—do you think you could write the equation for what you were just saying?

Note that Ms. Keith was not requesting an equation linked to the problem but rather a series of equations to reflect Hannah's strategy, step by step.

3 Discussion

In this article, we built on earlier work to present an emerging framework of teaching moves used in one-on-one interactions to support and extend children's mathematical thinking. The framework identifies categories of teaching moves, rather than specific comments or questions, because how teachers enact a category depends on the situation. We do not believe that there are any perfect comments or questions, and the same comment or question can be effective in one situation but not in another. The four major categories we discussed included ensuring that the child is making sense of the story problem, exploring details of the child's existing strategy, encouraging the child to consider other strategies, and inviting the child to generate symbolic notation. The category focused on exploring details of existing strategies was purposefully highlighted by the inclusion of five subcategories because teachers can build on children's thinking only after making sense of their existing thinking.

We illustrated the framework using examples from a case study of Ms. Keith's interactions with children engaged in solving fraction story problems in two settings. These data extend our earlier work on responsive-teaching moves from whole numbers to fractions and from one-on-

one interviews to class lessons. We realize the limitations of a single case study and acknowledge the need to further test the viability and comprehensiveness of the framework with additional teachers. However, we have begun using this framework to analyze other cases in our expert-teaching study, and preliminary results indicate the robustness of the framework categories (Jacobs et al. 2015). Further, we believe that Ms. Keith's case was an important starting point because Ms. Keith represents the group of teachers on the highly skilled end of the responsive-teaching continuum, and studying her moves enabled us to better understand how this type of teaching could be enacted and to collect clear illustrations of responsive-teaching moves.

As we worked on refining our framework of teaching moves, two additional issues were brought to the fore, which have potential for research and professional development: the uses of one-on-one interviews and the significance of circulating time during class lessons. We believe each merits further attention.

3.1 Uses of One-On-One Interviews

We recognize that teachers sometimes use one-on-one interviews as an instructional tool to explore their students' thinking in more depth than is possible in class settings. However, we are also excited about the additional potential of interviews, given our findings. Even with our small sample size, we were struck with how the types of moves Ms. Keith made in her interviews were the same types of moves that she used in the classroom, and, as reflected in our findings section, we found examples of each framework category in both settings. This consistency can be useful for both professional developers and researchers.

First, interviewing is a common activity in professional development because interviews isolate teacher-student conversations about mathematics from other aspects of class life, making them ideal for providing teachers with focused opportunities to practice their questioning skills and develop understanding of children's mathematical thinking (e.g., Clarke et al. 2011; Crespo and Nicol 2003). However, although many professional development projects, including our own, have embraced interviewing as a professional-development activity, identifying exactly what parts of expertise in interviews and class lessons overlap has been insufficiently researched. Results of this study provide an existence proof for the parallels between responsive-teaching moves in the two settings and, as such, reinforce the potential of interviews to support teacher learning by serving as what Grossman and colleagues (2009) called *approximations of practice.* Teachers use approximations of practice to engage in the work of teaching in manageable yet meaningful pieces. The more congruent to and integrated with the actual work of teaching these approximations are, the more authentic they are, and the more likely they are to help teachers develop skills that are usable in their daily work. Our data indicate a high degree of authenticity between teaching moves in interviews and class lessons.

Second, given these authentic links, researchers may find that, at times, analyzing teachers' interviews provides a more feasible means of assessing teachers' practice than analyzing class lessons. We are not claiming that all teachers who show expertise while interviewing will be able to show the same expertise in a class lesson, where they often must not only interact with one child but also manage many children simultaneously. Nonetheless, our findings provide support for the idea that the expertise needed for responsive interviewing is necessary, but likely not sufficient, for responsive teaching in class lessons; teachers who cannot be responsive to individual children's thinking during interviews will most likely be unable to do so in a class setting. Thus, researchers may be able to use these interviews as a window into the upper bound of teachers' expertise in an important segment of their practice.

3.2 Significance of Circulating Time During Class Lessons

Our analysis focused on the part of class lessons in which teachers circulate and engage in one-on-one conversations with children during problem solving, and we believe that this part of teaching is often underappreciated and deserves more attention in both research and professional development. Ms. Keith spent one third to one half of each lesson in these conversations, and virtually all of them were organized around responding to children's mathematical thinking. During these moments, her moves were closely calibrated to children's thinking; they did not require children to abandon what they were thinking or to take steps they were not ready to take. If building new understanding consists of constructing a network of connections related directly to what is already understood (Carpenter and Lehrer 1999), then these kinds of conversations provide continual opportunities for children to advance their mathematical thinking.

Circulating has often been viewed as an ideal time for teachers to monitor students' ideas in preparation for whole-group discussions (Stein et al. 2008), and we would agree. However, we believe that these one-on-one conversations also provide important learning opportunities in their own right, and teachers may be missing opportunities to advance students' thinking during these moments. We want to be clear that we also value the power of whole-group discussions (Chapin et al. 2009; Franke et al. 2009), and we note that Ms. Keith included whole-group discussion in each lesson. Nonetheless, we believe that the field would benefit from considering teachers' circulating time more broadly.

3.3 A Final Caveat

In this article, we put forth a framework of teaching moves as a tool for capturing and encouraging responsive teaching. However, because learning is difficult, and having one's thinking probed and pushed can be uncomfortable, we issue a final caveat—for the framework to be effective, strong teacher-student relationships need to underlie responsive interactions.

A caring and respectful stance was evident throughout Ms. Keith's interviews and class lessons in her comments, tone, and body language. She upheld this stance even when asking probing questions or pushing children to think in new ways by always ensuring that children maintained control of their problem-solving processes (e.g., "If *you* think about that, could that help *you*?"). Ms. Keith also regularly asked children to reflect on their problem-solving processes in terms of both the mathematics and how the problem solving felt to them. Sometimes she asked children to reflect on a specific task (e.g., "How did these [story problems] feel to you?") and other times, she requested reflection on their development as mathematical thinkers (e.g., "What do you think about how you've changed in how you work with math?").

We believe that all of these—seemingly small—efforts were essential in setting the stage for Ms. Keith's responsive moves to be productive in advancing children's thinking. Thus, we conclude with an appreciation for the foundational role of teacher-student relationships in teaching that is responsive to children's mathematical thinking.

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